Estimation of Bifurcation Points in Duffing Equation with Double Minimum Potential

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Summary : Bifurcation phenomena are investigated in Duffing equation with double minimum potential under weak external force. They arise when the oscillation confined within one side well extends over two wells, and cease when the oscillation can not feel the double minimum nature of the original potential for the sake of its large amplitude with the increase of the external force. The manner of the bifurcation is very sensitive to the frequency of the external force. We succeeded in roughly estimating the critical values of the external force at which the bifurcations occur and cease, using harmonic approximation of the original double minimum potential and assuming the presence of fundamental oscillation.

Key Words : Duffing equation, Double minimum potential, Bifurcation point

1. Introduction

The forced Duffing equation[1] describing a motion of classical particle with damping in an anharmonic potential, $V(x)=(\alpha x^4-2\beta x^2)/4(\alpha>0)$, under periodic external force,

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}+\alpha x^3-\beta x=E\cos\nu t, \qquad (1)$$

has been well investigated, since it shows various interesting phenomena associated with a nonlinearity of the potential despite of its simple structure[2,3,4,5,6]. Novak and Frehlich suggested a mechanism of period-doubling to lie in a parametric excitation of subharmonics, and compared their theory with experiment[3]. The agreement was satisfactory. Sato, Sano and Sawada confirmed a scaling property in a global bifurcation set[6]. In both cases, $\beta<0$ or $\beta=0$, that is, the potential is single minimum. In such a case, aperiodic phenomena appear for sufficiently intense external force associated with the nonlinearity of the potential proportional to x^4. The bifurcation phenomena reported since now have been for such case.

On the other hand, if $\beta>0$, the potential is double minimum. Characteristic features of the motion for the double minimum potential should appear when the particle motion almost confined in one side well under weak external force transforms into that extending over two wells with the increase of the external force. In other words, they should appear when the energy of the particle measured from the bottom of the potential is smaller than or is comparable with the depth of the potential well. But the investigations on such standpoint have not been seen to the author's knowledge.

The purpose of this article is to investigate the characteristic features of the motion reflecting the "double minimum" potential. The results of numerical calculations show that the bifurcation scheme is sensitive to a parameter describing the potential and to a frequency of the external force. But we report only the outline of the scheme and do not enter into the details of them. We focus our attention to estimate the critical external forces at which the bifurcation arises. In the next section, the numerical results are mentioned in detail. In section 3 elementary derivations will be tried of three critical external forces at which each bifurcation arises using appropriate approximations. The last section is devoted to summary and concluding remarks.

2. Numerical Results

In eq.(1) we choose large damping constant, $k=1$, `since complicated regions exist if k is smaller than 0.6 as shown in Fig. 3 of ref. 6. The parameter, α, decides a curvature of the potential proportional to x^4, and is not so essential. We take $\alpha=2$. Equation (1) is analyzed through second order Runge-Kutta method. The initial condition does not affect the behavior of the stationary solution, since eq.(1) contains large damping term. In the following it is taken as $x(0)=2$ and $\dot{x}(0)=1$. To exclude the initial transient part from the solution, the data within 2×10^4 steps are omitted. The behaviors of the solution largely depend on two parameters, β and ν. We fix $\beta=1.2$ and take ν as a parameter. Hereafter we use E as a strength of the external force.

Typical patterns of the orbits are given in Fig.1 for $\nu=1$. For small E, simple fundamental oscillation mode is seen in Fig.1

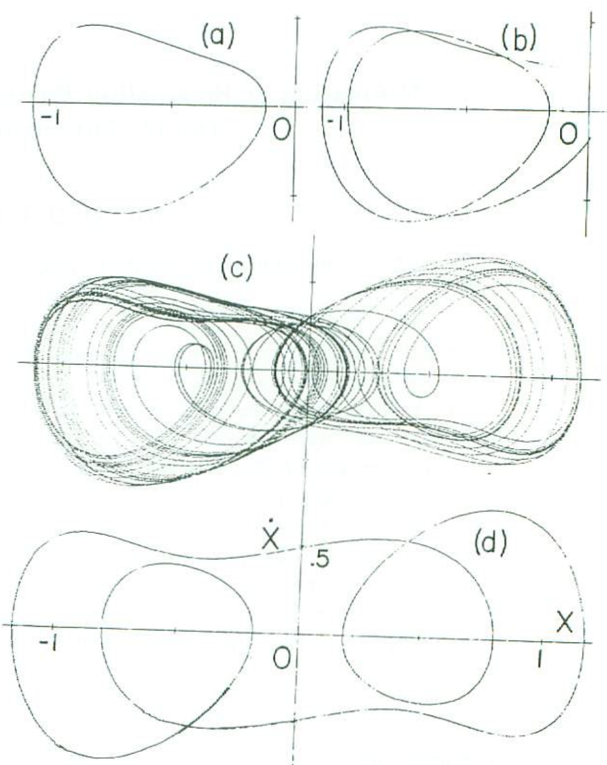

Fig.1. Typical patterns of the orbits when $\nu=1$. E is taken to be 0.53, 0.58, 0.64 and 0.78 for (a), (b), (c) and (d), respectively.

(a), but the period-doubling takes plae for $E=0.58$(Fig.1(b)). In both cases the motion is seen to be confined in one side of two wells. For $E=0.64$(Fig.1(c)) the motion extends over two wells and is chaotic. Fig.1(d) shows the orbit in window region, $E=0.78$. To see the E-dependence of the solution in detail, we observe the points at which the orbit passes through the x-axis, that is, the points satisfying $\dot{x}(t)=0$. The results for $\nu=0.8$, 1.0, 1.2 and 1.4 are represented in Figs. 2 and 3. We investigate the region of ν ranging from 0.1 to 2.2. For $\nu<0.3$ bifurcation phenomena can not be observed, and

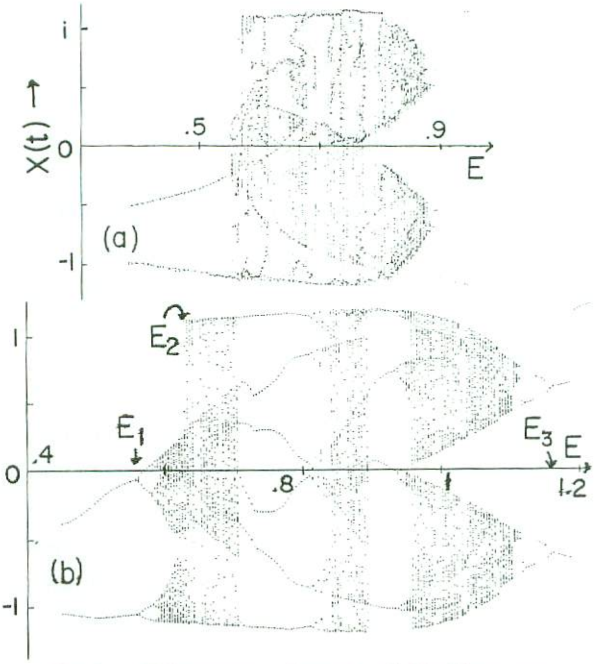

Fig.2. Bifurcation digram of Duffing equation with double minimum potential. The point at which the orbit passes through the x-axis is plotted against E when $\nu=0.8$ and 1 in (a) and (b).

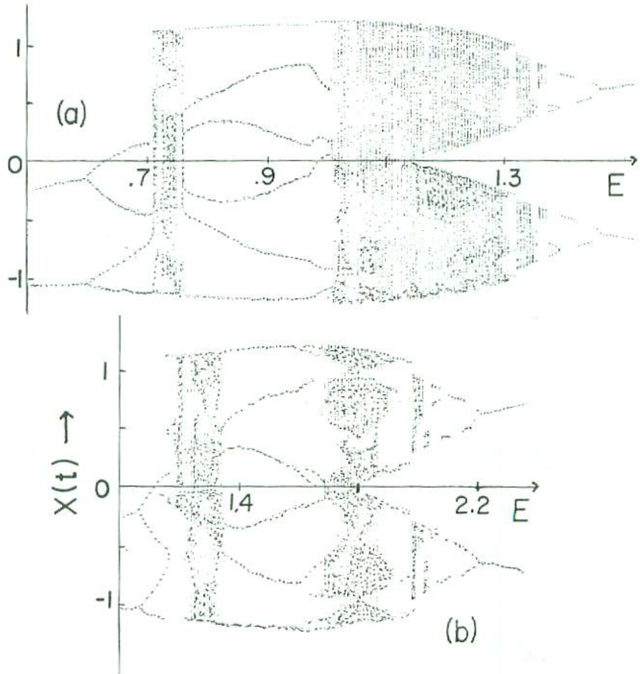

Fig.3. Bifurcation diagram of Duffing equation. The points at which the orbit passes through the x-axis are plotted against E when $\nu=1.2$ and 1.4 in (a) and (b), respectively.

we do not discuss this region. In any cases simple periodic orbit (fundamental oscillation mode) exists for $E\leq 0.5$, and is seen to be confined in one side of the two wells. Increase of E induces the bifurcation of the orbit, the manner of which depends on ν. For $\nu<0.8$ the chaotic orbit appears suddenly as shown in Fig.2(a). For $\nu>0.9$ the period-doubling takes place in the first place as shown in Figs.2(b) and 3. Accumulation of the period-doubling can not be observed. We call the value of E as E_1 at which the simple orbit bifurcates at first. E_1 is 0.5589 when $\nu=1$. After a finite sequence of period-doubling, the solution becomes chaotic for $\nu>0.9$. The motion is still confined in one side of the potential wells for $E\leq E_1$. More increase of E from E_1 makes the motion extend over two wells at $E=E_2$ which is 0.6307 for $\nu=1$. When $\nu<0.8$, chaotic region and window region appear successively. After that two chaotic bands generate, and they transform again into simple orbit of fundamental oscillation (Fig. 2(a)) as E increases from E_3. While for $\nu>0.9$, two chaotic bands transform into simple periodic orbit accompanying the inverse process of period-doubling after appearences of one or two chaotic and window regions as shown in Fig.2(b) and 3. We call the value of E as E_3 at which the bifurcation ceases and fundamental oscillation gains its stability again. E_3 is 1.162 for $\nu=1$. Three characteristic values, E_1, E_2 and E_3 are indicated in Fig.2(b).

The region in $E-\nu$ plane where bifurcation occurs is represented in Fig.4 by cross hatch-ed region. The lowest limits of E and ν are 0.55 and 0.3, respectively. The fact that the region of bifurcation is restricted in a finite range of E for fixed ν, is characteristic point that distinguishes the bifurcation phenomena observed here from those of global bifurcation set reported since now on Duffing equation. Above fact clearly shows that the bifurcation of the solution originates from the motion feeling the non-linearity of the double minimum potential.

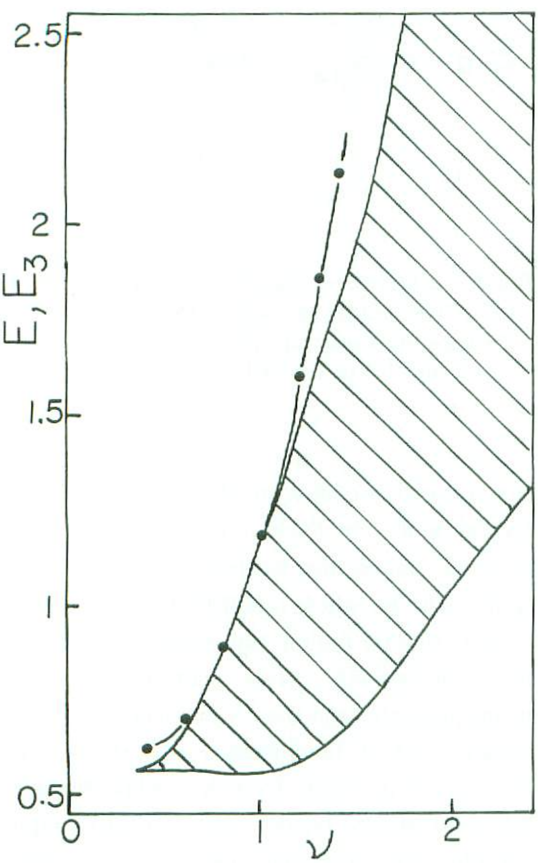

Fig.4. The region of bifurcation in $E-\nu$ plane. The cross hatched region is this. Solid circles show the theoretically estimated values of E_2.

Typical example of the Poincaré map is given in Fig.5 in chaotic region of $\nu=1$ at $E=0.64$ and 0.84. It is constructed as follows: let x_1 be the point satisfying $\dot{x}(t)=0$ first, and x_2 be such point next time. We obtain x_1, x_2, $\cdots$ successivly, and plot them in (x_n, x_{n+1})-plane. Clear one dimensional map is obtained, however we do not discuss the details of them here. For $E \gg 1$, well known folding of the orbit and the global bifurcation set are observed.

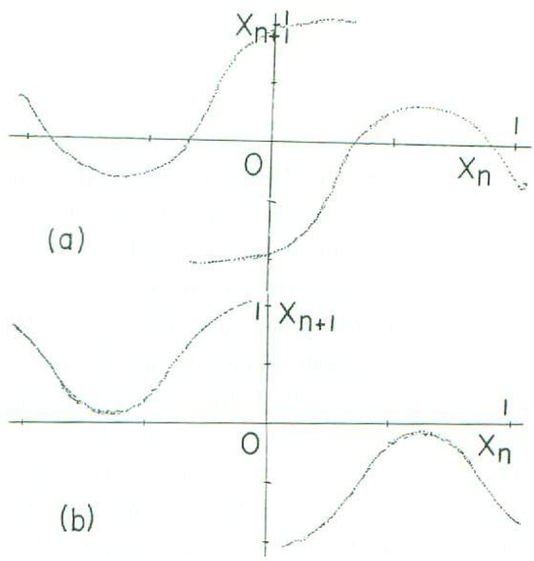

Fig.5. Poincare maps for $E=0.64$((a)) and 0.84((b)) when $\nu=1$.

3. Estimation of E_1, E_2 and E_3

3-1 Estimaion of E_1 and E_2

The potential, $V(x)=(\alpha x^4-2\beta x^2)/4$, has depth of $-\beta^2/4\alpha$ at $x_0=\pm\sqrt{\beta/\alpha}$. When the external force is weak enough, the particle is considered to be localized near the bottom of one side potential well. Thus letting $x(t)=x_0+\xi(t)$, $|\xi(t)|\ll 1$, and linearizing eq. (1), we obtain,

$$\frac{d^2\xi}{dt}+k\frac{d\xi}{dt}+2\beta\xi\cong E\cos\nu t. \qquad (2)$$

Here $V(x)$ is approximated to be harmonic as $V(\xi)\cong-\beta^2/4\alpha+\beta\xi^2$. Equation (2) can be solved elementary. The stationary solution is given as,

$$\begin{aligned}\xi(t)=&\frac{2E}{\omega}\frac{1}{k^2+4(\nu-\omega)^2}\{\frac{k}{2}\sin(2\omega-\nu)t\\&+(\nu-\omega)\cos(2\omega-\nu)t\}\\&+\frac{2E}{\omega}\frac{1}{k^2+4(\nu+\omega)^2}\{-\frac{k}{2}\sin\nu t\\&+(\nu+\omega)\cos\nu t\},\end{aligned} \qquad (3)$$

where $\omega=\sqrt{8\beta-k^2/2}$. Using $\xi(t)$ of eq.(3), we calculate the average kinetic and potential energies of the particle moving in harmonic potential, K and U. Since the mass and the force constant of the approximated potential are 1 and 2β, respectively, K and U are estimated as follows:

$$\begin{aligned}K\cong&<\frac{1}{2}\xi^2>=\frac{E^2}{4\omega^2}\{\frac{\nu^2}{k^2+4(\nu+\omega)^2}\\&+\frac{(2\omega-\nu)^2}{k^2+4(\nu-\omega)^2}\},\end{aligned} \qquad (4)$$

$$\begin{aligned}U\cong&<\frac{1}{2}2\beta x^2>=\frac{\beta E^2}{2\omega^2}\{\frac{1}{k^2+4(\nu+\omega)^2}\\&+\frac{1}{k^2+4(\nu-\omega)^2}\}.\end{aligned} \qquad (5)$$

Here $<\cdots\cdots>$ denotes the average over a time of common multiple of $2\pi/\nu$ and $2\pi/|2\omega-\nu|$. Let us discuss the relation between K, U and the depth of the potential well, $D=\beta^2/4\alpha$. If the total energy of the particle, K+U, is much smaller than D, the particle would behave harmonically near the bottom of the potential. But K and U increases with E. When $K+U\cong D$, the particle should feel the nonlinearity of the potential. Thus if a condition for period-doubling exists, it should appear at E_1 satisfying $K+U=D$. The condition, $K=D$, will be reached with the more increase of E. What happens in this case ? The particle motion will extend over two wells, since the kinetic energy prevails the potential barrier between two wells. Accordingly the condition, K=D should give E_2.

The estimated values of E_1 and E_2 are compared with those obtained numerically in Fig. 6. Above estimations can not at all be applied for $\nu<1$, while they give rough agreement with data for $\nu>1$. The exact coincidences of E_1 at $\nu=1.2$ and of E_2 at $\nu=1.4$ are of course accidental. The relative error is within 20% between $\nu=1.1$ and 2.3, for example, when $\nu=1.8$, estimated values of E_1 and E_2 are 0.760 and 1.279, while the numerically obtained values are 0.901 and 1.146, respectively. The relative errors are 8% and 10%. Considering the method of estimation to be very elementary, the agreement is satisfactory The difficulty lies in the failure of the estimation for $\nu<1$

3-2 Estimation of E_3

The bifurcation observed in this study may cease when the motion extends over a wide region with the increase of E, and the particle can not already feel that the potential is double minimum. In such a case the har-

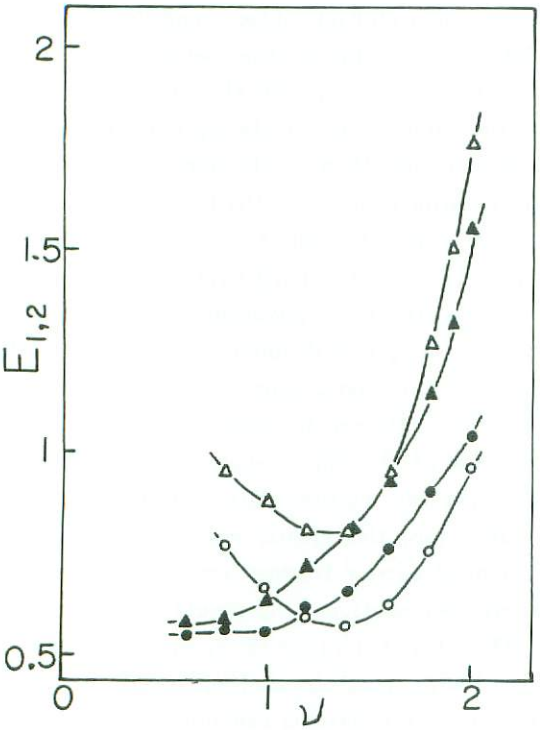

Fig.6. Comparison of estimated values of E_1 and E_2 with numerically obtained values. Solid and open circles show E_1 obtained numerically and theoretically, respectivly. Similarly solid and open triangles represent E_2 obtained theoretically and numerically.

monic potential approximation done in the preceeding paragraph breaks down, and we must treat eq.(1) directly. According to the treatment described in usual textbook[1], we assume a fundamental oscillation solution to eq.(1) as $x(t) = a\cos\nu t + b\sin\nu t$. Substituting it into eq.(1) and equating the terms of $\cos\nu t$ and $\sin\nu t$, we obtain the following relation:

$$\{k^2\nu^2 + (\frac{3}{4}\alpha r^2 - \beta - \nu^2)\}r^2 = E^2, \qquad (6)$$

where $r^2 = a^2 + b^2$. If the amplitude, r be larger than the width of the potential well, $\sqrt{2\beta/\alpha}$, the particle can not already feel the potential to be double minimum. Thus we obtain E_3 from eq. (6) by letting $r = \sqrt{2\beta/\alpha}$, that is,

$$E_3 = \{k^2\nu^2 + (\beta/2 - \nu^2)\}^{1/2}(2\beta/\alpha)^{1/2}. \qquad (7)$$

Estimated value of E_3 is represented in Fig. 4. It fairly reproduces the numerical data in the region of $\nu \leq 1.8$, however the discrepancy becomes remarkable for $\nu > 1.8$. The reason is not clear yet.

4. Summary and Concluding Remarks

The bifurcation phenomena of Duffing equation have been reported by many authors, however they are concerned with the nonlinearity of the potential proportional to x^4, and they are observed at sufficiently intense external force. When the potential is double minimum, the nonlinearity should appear at weak enough external force, since the transformation of the solution exist from that confined within one side of the two wells to that extending over two wells. It is confirmed numerically that the bifurcation takes place even at sufficiently weak external force, where in usual analyses fundamental oscillation mode is assumed. The bifurcation ceases when the amplitude becomes larger than the width of the potential well and the motion can not feel the double minimum nature of the potential.

Accordingly the behaviours of the bifurcation have different properties from those reported since now. The first peculiar point is that the bifurcation is sensitive to ν. The particle moves in nearly harmonic potential under weak external force, thus the frequency intrinsic to the system exists, from which it is reasonable that the bifurcation depends on ν. Contrary to the case of this study, the manner of bifurcations reported since now does not depend on ν essentially, since such intrinsic frequency does not exist. Secondly the region where the bifurcations are observed, is restricted in a finite range of the external force The bifurcation phenomena reported since now can generate in an infinite range of the external force, because they are related to the nonlinearity of the potential proportional to x^4. On the other oand, the bifurcations observed in this study are concerned with the nonlinearity that the potential is double minimum.

We succeeded in roughly estimating the characteristic external forces, E_1, E_2 and E_3 at which the first bifurcation occurs, the chaotic motion extends over two potential wells and the bifurcation ceases, respectively. E_1 and E_2 are determined by comparing the depth of the potential well with the particle energy calculated from harmonic approximation of the double minimum potential. E_3 is decided by equating the width of the potential well to the amplitude of the motion in fundamental oscillation mode. To our regret, the agreements with numerical data are unsatisfactory in the region of $\nu<1$ concerning E_1 and E_2 and in the region of $\nu>1.8$ for E_3. Although some corrections, such as renormalization of the frequency, seems to be neccessary in the above region, the estimation gives satisfactory result near $\nu=1$. To the best of our knowledge, this is the first trial to obtain the characteristic external forces at which the bifurcation starts or ceases, in double minimum potential.

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2重井戸型ポテンシャルを有する ダッフィン方程式における分岐点の評価

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要旨：ダブルミニマムポテンシャルを有する Duffing 方程式の解は，外力が弱く振動がポテンシャルの底に閉じ込められている時は調和的であるが，外力が増し非線型性が感じられるようになると分岐現象が発生する．外力が更に強くなり，解の振巾が増加し，ポテンシャルがダブルミニマムであることが感じられなくと分岐は終了する．以上の事実を数値実験により確認した．分岐が発生する外力の値はもとのポテンシャルを調和近似し，振動のエネルギーを求め，それをポテンシャルの井戸の深さと比較することにより求めた．分岐の終了は，基本調波振動を仮定し，その振巾が井戸の巾と等しくなるという条件で定めた．定性的には納得できるが，定量的に数値解と比較するには不充分な近似であることが解った．